

# Initial Conditions and Entanglement Sudden Death

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We report results bearing on the behavior of non-local decoherence and its potential for being managed or even controlled. The decoherence process known as entanglement sudden death (ESD) can drive prepared entanglement to zero at the same time that local coherences and fidelity remain non-zero. For a generic ESD-susceptible Bell superposition state, we provide rules restricting the occurrence and timing of ESD, amounting to management tools over a continuous variation of initial conditions. These depend on only three parameters: initial purity, entanglement and excitation. Knowledge or control of initial phases is not needed.

## I. INTRODUCTION

A continuously interacting background environment tends to destroy quantum state coherence. Quantum non-separability (entanglement), which plays a key role in various quantum computing and quantum communication proposals [1], is the best-known example of non-local coherence. Regarding separability, it is known that there is set of finite measure of separable states. If the steady state solution of a master equation for an initially entangled state lies on the interior of this set, then the transition from an entangled to a separable state has to occur after only a finite time of evolution. A version of these facts is discussed in specific terms in connection with an event which is called ESD (early-stage decoherence or more commonly entanglement sudden death) by Yu and Eberly [2] and by Al Qasimi and James [3]. The inevitability of ESD for a class of initial states does not, however, make available the actual time when ESD will occur. Knowledge of the ESD time allows one, in at least some situations, to undertake preparations to avoid or delay its onset. In the following we address this question.

Studies have been made to examine the ways in which ESD depends on dissipation mechanisms (e.g., amplitude damping [4], phase damping [5], and others [6]), and/or on interaction structures (e.g., interacting [7] or non-interacting [8] entangled partners, common [9] or independent [10] reservoirs), etc. Most of these factors are difficult to control in practice. In this Letter we extend the study of ESD control in a straightforward way by systematically deriving connections between ESD and initial conditions, about which only isolated facts are noted, e.g., in the study of ESD for Werner-like [11] and X-type [12] of initial states. We focus on the consequences of the initial preparation of entanglement, purity, and double excitation probability. Their initial values may be critical for an experiment's design and also for avoiding noise-induced ESD during the experimental time interval, but these initial values are not always easy to establish precisely. However, for the generic case of exposure to amplitude noise, one can identify initial-value phase spaces which contain finite zones of safe prepa-

ration. That is, within the well-defined boundaries of these zones the state evolution is guaranteed not to be ESD-susceptible. To our knowledge, this is the first indication that such phase boundaries exist for any specific dissipation mechanism, and we provide formulas for the boundary curves.

Additionally, analysis of the ESD-susceptible phases allows us to determine another feature of ESD that has remained generically unknown, namely the precise time when ESD will occur, if it is to occur at all. The ESD onset time is found to be very sensitive to the initial purity in the vicinity of critical purity points. Especially for initial states with very high or low entanglement, we show that a small amount of state impurity will accelerate the ESD process dramatically. Among potential advantages in practice, the conditions we have found allow protection from ESD and may be useful in guiding experiments toward preparation of states that are immune to ESD or have relatively longer ESD times.

## II. PREPARED INITIAL STATE

As an example we can consider the not quite perfect preparation of Bell states  $|\Phi^+\rangle$  and  $|\Phi^-\rangle$ , which are known to be vulnerable to ESD in any combination. We suppose that a quantum computation task aims to use a generic combination such as

$$|\Phi_{AB}\rangle = \cos\theta|e\rangle_A|e\rangle_B + \sin\theta|g\rangle_A|g\rangle_B, \quad (1)$$

where  $|e\rangle_A$  and  $|g\rangle_A$  are orthogonal states of qubit  $A$ , etc., and may be interpreted as excited and ground states. In this state the degree of entanglement between the two qubits, as measured by concurrence [13], is  $C = \sin 2\theta$ .

In reality, in preparation, a state will deviate from the ideal target state. To sketch how this might occur we can begin with the state after its preparation. It is coupled to an environment that will cause post-preparation decoherence, and also still entangled with marginal entities denoted by  $M$ , i.e., forces, fields and objects that may participate in the preparation phase but then cease

interaction (see [14, 15]). Thus we write:

$$|\Phi(0)\rangle = \left[ \cos\theta|e\rangle_A|e\rangle_B|m_1\rangle + \sin\theta|g\rangle_A|g\rangle_B|m_2\rangle \right] \otimes |\phi_0\rangle_a|\phi_0\rangle_b. \quad (2)$$

Here  $\theta$  determines the degree of excitation of the two-party state via  $\cos^2\theta = \rho_{11}$ ,  $\sin^2\theta = \rho_{44}$ , and  $|\phi_0\rangle_a$  and  $|\phi_0\rangle_b$  are the normalized initial states (usually ground states) of environmental reservoirs  $a$  and  $b$  respectively, and  $|m_1\rangle$  and  $|m_2\rangle$  are normalized states of the marginal system  $M$ . Imperfect control of the preparation leads to a possibly mixed rather than pure initial state via the relation  $\sin\theta\cos\theta\langle m_2|m_1\rangle = \rho_{14} \leq \sqrt{\rho_{11}\rho_{44}}$ . Here  $\rho_{ij}$  are the matrix elements of the initial two-qubit reduced density matrix

$$\rho_{AB}(0) = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \quad (3)$$

### III. ENTANGLEMENT EVOLUTION

We next consider the dynamics of the two-qubit entanglement to see how it is affected by the initial state specification, and expose each qubit to the same amplitude damping process [1]:

$$\begin{aligned} |e\rangle_A|\phi_0\rangle_a &\rightarrow \sqrt{q_a}|e\rangle_A|\phi_0\rangle_a + \sqrt{p_a}|g\rangle_A|\phi_1\rangle_a \\ |g\rangle_A|\phi_0\rangle_a &\rightarrow |g\rangle_A|\phi_0\rangle_a, \end{aligned} \quad (4)$$

where  $|\phi_1\rangle_a$  is another environmental reservoir state with  ${}_a\langle\phi_1|\phi_0\rangle_a = 0$ , and  $q_a$  is the probability that qubit  $A$  remains in its excited state. For the simplest illustration we assume exponential decay and take  $q_a(t) = \exp(-\Gamma t) \equiv \exp(-\tau)$ . Then the excitation-transfer probability  $p_a = 1 - q_a$  will grow from 0 to 1 irreversibly with increasing dimensionless time  $\tau$ . Qubit  $B$  and reservoir  $b$  are characterized similarly as (4) with decay probability  $q_b$ . Of course this is more specialized than needed - the rates of change  $\Gamma$  need not be the same for both qubits, and the amplitude channel can also be used to characterize non-dissipative evolutions such as produced by the XY spin interaction [16] or the Jaynes-Cummings interaction [17].

Amplitude damping takes the initial state (2) to the time-dependent state

$$\begin{aligned} |\Phi(t)\rangle &= \cos\theta\sqrt{q_aq_b}|ee\rangle|\phi_0\rangle_a|\phi_0\rangle_b|m_1(t)\rangle \\ &+ \cos\theta\sqrt{q_ap_b}|eg\rangle|\phi_0\rangle_a|\phi_1\rangle_b|m_1(t)\rangle \\ &+ \cos\theta\sqrt{p_aq_b}|ge\rangle|\phi_1\rangle_a|\phi_0\rangle_b|m_1(t)\rangle \\ &+ \cos\theta\sqrt{p_ap_b}|gg\rangle|\phi_1\rangle_a|\phi_1\rangle_b|m_1(t)\rangle \\ &+ \sin\theta|gg\rangle|\phi_0\rangle_a|\phi_0\rangle_b|m_2(t)\rangle. \end{aligned} \quad (5)$$

Here because of the fact that the marginal entity has ceased interaction with the qubit system after  $t = 0$ ,

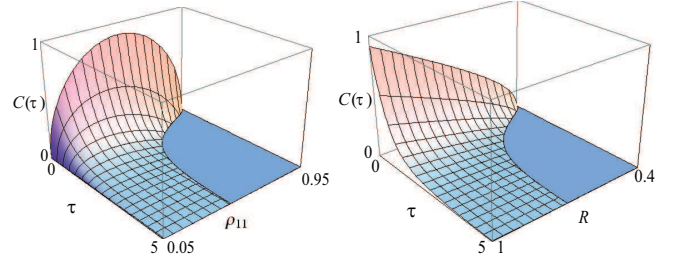


FIG. 1: Time-dependent surfaces showing entanglement evolution parameterized in two ways: (left)  $C$  vs.  $\tau$  for a range of prepared double excitation values, and fixed purity; and (right)  $C$  vs.  $\tau$  for a range of purities for fixed double excitation. The symbols are defined in the text. In both plots the solid-color base-plane shows the region where  $C = 0$ , i.e., where ESD has already occurred. It is clear that ESD-free regions are accessible, but the extreme time-sensitivity of the ESD boundaries is not obvious here.

the time dependent marginal states preserve the overlap relation  $\langle m_2(t)|m_1(t)\rangle = \langle m_2|m_1\rangle$ . Then the two-qubit time-dependent reduced density matrix is obtained by tracing off the reservoir and marginal states, yielding:

$$\rho_{AB}(t) = \begin{pmatrix} \rho_{11}q_aq_b & 0 & 0 & \rho_{14}\sqrt{q_aq_b} \\ 0 & \rho_{11}q_ap_b & 0 & 0 \\ 0 & 0 & \rho_{11}p_aq_b & 0 \\ \rho_{41}\sqrt{q_aq_b} & 0 & 0 & \rho_{11}p_ap_b + \rho_{44} \end{pmatrix}. \quad (6)$$

The two-party entanglement (concurrence [13]) obeys the X-state formula [18]:

$$C(\tau) = 2\sqrt{q_a(\tau)q_b(\tau)} \times \max\{0, |\rho_{14}| - \rho_{11}\sqrt{p_a(\tau)p_b(\tau)}\}, \quad (7)$$

which is graphed in two ways in Fig. 1.

Three parameters associated with the state mentioned above, that are presumably under good if not perfect preparation control [19], will now be selected to guide examination of evolution. These parameters are  $C_0$ ,  $\rho_{11}$  and  $R$ , namely the initial entanglement (concurrence), initial excited state population, and a normalized value of purity. The true initial purity is given by

$$\begin{aligned} P &= 1 - 2(\rho_{11}\rho_{44} - |\rho_{14}|^2) \\ &= 1 - 2\rho_{11}(1 - \rho_{11}) + C_0^2/2, \end{aligned} \quad (8)$$

and the normalized version is defined as

$$R = \sqrt{2P - 1}, \quad (9)$$

which takes values conveniently between 0 and 1.

### IV. ESD SUSCEPTIBILITY PHASE DIAGRAMS

The plots in Fig. 1 are only for a specific category of two-party states, but are systematically exhaustive in

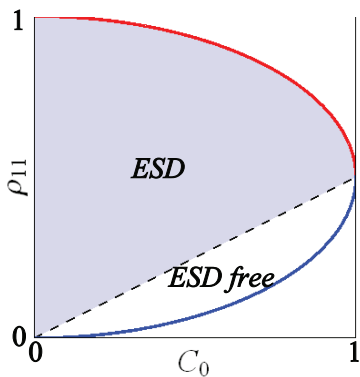


FIG. 2: Entanglement phase diagrams of the initial two-qubit state in terms of the initial concurrence  $C_0$  versus the double excitation number  $\rho_{11}$ . The ESD phase (shaded area) and ESD-free phase (unshaded area) are separated by the dashed critical line  $\rho_{11} = C_0/2$ .

the sense that any arbitrary combination of the  $\Phi^\pm$  Bell states is included, as well as any initial purity, excitation strength, and concurrence. It turns out that their ESD boundaries are directly analyzable, as follows.

First we use (6) and (7) at  $\tau = 0$ , where  $q = 1$  and  $p = 0$ , to characterize the entire physical domain in terms of  $\rho_{11}$  and  $C_0$ . Non-negativity of  $\rho$  requires the value of  $\rho_{11}$  to be bounded between  $\rho_{11}^{\max}$  and  $\rho_{11}^{\min}$ , i.e.,  $[1 \pm \sqrt{1 - C_0^2}]/2$ , for any fixed  $C_0$ . From the fact that  $C_0 \leq 1$ , one immediately has

$$\rho_{11}^{\min} \leq C_0/2 \leq \rho_{11}^{\max}. \quad (10)$$

Within this domain, as  $\tau \rightarrow \infty$ , both  $p_a$  and  $p_b$  and their product grow from 0 to 1. Then it is easy to identify within the physical domain the boundaries between ESD and non-ESD phases. The condition  $C(\tau) = 0$  tells us that ESD occurs wherever

$$\sqrt{p_a p_b} \geq \frac{|\rho_{14}|}{\rho_{11}} = \frac{C_0}{2\rho_{11}}. \quad (11)$$

From the fact that  $\sqrt{p_a p_b} \leq 1$ , one immediately notes that ESD is restricted to the domain where the initial double excitation number is greater than half of the initial concurrence, i.e.,  $C_0/2 < \rho_{11} \leq \rho_{11}^{\max}$ . The opposite region  $\rho_{11}^{\min} \leq \rho_{11} \leq C_0/2$  is ESD-free. This leads to the phase diagram in Fig. 2, which locates the ESD and ESD-free phases in terms of initial concurrence  $C_0$  and the double excitation probability  $\rho_{11}$ . The solid (in color, red and blue) boundary lines represent the values  $\rho_{11}^{\max}$  and  $\rho_{11}^{\min}$  as a function of  $C_0$ . The dashed line  $\rho_{11} = C_0/2$  is the critical phase boundary where the state crosses from the ESD-free phase to the ESD-inevitable phase.

An important implication of this analysis is that one needs to prepare the state with low double excitation,  $\rho_{11} \leq C_0/2 \leq 1/2$ , in order to avoid ESD. Therefore we will focus on this region of greatest interest and assume  $\rho_{11} \leq 1/2$  in the following discussion.

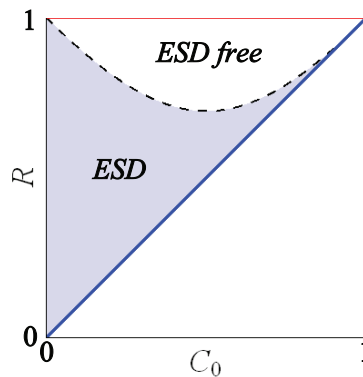


FIG. 3: Entanglement phase diagram in terms of the initial concurrence  $C_0$  versus the initial normalized purity  $R$ . The ESD phase (shaded area) and ESD-free phase (unshaded area) are separated by the dashed critical line  $R_c = \sqrt{2C_0^2 - 2C_0 + 1}$ .

We now go a further step by employing the fact that the initial concurrence  $C_0$  and the double excitation probability  $\rho_{11}$  are simply related to the initial state purity  $P$  as in (8). Then the ESD-onset condition (11) can be rewritten in terms of the initial concurrence and normalized purity  $R$  as follows

$$\sqrt{p_a p_b} \geq \frac{C_0}{1 - \sqrt{R^2 - C_0^2}}, \quad (12)$$

where the non-negativity of the initial density matrix now requires that the normalized purity  $R$  obeys  $R_{\min} = C_0$  and  $R_{\max} = 1$  for any fixed  $C_0$ . Fig. 3 locates the ESD and ESD-free phases in  $C_0$ - $R$  space. The (in color, red and blue) solid lines represent the values  $R_{\max}$  and  $R_{\min}$  as functions of  $C_0$ . The dashed line is defined by

$$R_c = \sqrt{2C_0^2 - 2C_0 + 1} \quad (13)$$

which specifies the critical purity, as a function of  $C_0$ , where the state crosses from the ESD-free phase to the ESD-inevitable phase.

One sees that the states with very high or very low initial concurrences are quite fragile and require very high purity in order to avoid ESD. States with initial concurrence around 0.5 are the most robust, because they tolerate the widest range  $[\sqrt{1/2}, 1]$  of variation of  $R$  or, in other words, have higher tolerance of preparation errors (permitting lower initial purity) before breaking down to be ESD-susceptible. This may have application in preparing ESD-free states.

## V. ESD ONSET TIME

Although the phase boundaries are now located, there remains a question of near-boundary values of concurrence. In the ESD-inevitable regions the onset of ESD

is not instantaneous so temporary preservation of entanglement can be obtained if the parameter region close enough to the phase boundary for ESD onset can be well defined.

Within an ESD-susceptible zone we denote by  $T$  the time when ESD occurs and entanglement permanently vanishes, i.e., the ESD onset time. That is,  $T$  is the value of  $\tau$  when relation (12) is an equality. With  $p_a = p_b = 1 - e^{-T}$ , the onset time is then given by

$$T = \ln \left[ \frac{1 - \sqrt{R^2 - C_0^2}}{1 - C_0 - \sqrt{R^2 - C_0^2}} \right], \quad (14)$$

which is displayed in Fig. 4 as a function of both  $R$  and  $C_0$ . One immediately sees from Fig. 4 that for any fixed initial concurrence  $C_0 \in [0, 1]$ , the ESD time is very sensitive to the initial purity in the vicinity of its critical values  $R_c$ . Especially for the states having very high or very low initial concurrences, a small departure of the initial state from  $R = 1$  will accelerate the ESD process dramatically.

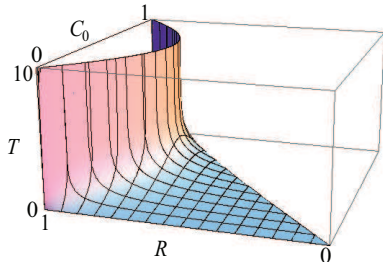


FIG. 4: Entanglement sudden death time as a function of the initial entanglement  $C_0$  and initial normalized purity  $R$ . The steepness of the boundary indicates existence of a narrow zone of “safe” state preparation, even though susceptible to ESD.

An important practical point is that we can talk about three zones with clear meaning. One is the zone of infinite onset time, where ESD cannot occur, and one is the zone where ESD occurs inevitably and very rapidly. A more interesting zone is the one where the onset time for ESD is delayed in a definite way. A practical issue could be that we can tolerate one dissipative lifetime of decay, but only if we can be assured that ESD will not occur. This means that we need to know the tolerable equivalent parameter range for the initial preparation. To illustrate this case, we impose on Fig. 3 another boundary line, located at  $T = 1$ , and show the result in Fig. 5. The yellow zone shows how the parameter range available to the preparation process can be expanded into the ESD-susceptible phase, tolerating some entanglement dissipation while remaining certain to avoid ESD up to the predetermined time  $T = 1$ .

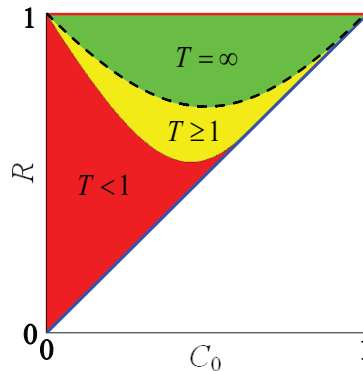


FIG. 5: ESD occurrence is identified here by time to onset of ESD. The graph can be read as a traffic signal, where green is a safe zone with no possibility of ESD, and red is the no-go zone where ESD is sure to occur quickly (within one lifetime of the dissipative process or less), and the yellow zone is one where ESD is certain to occur but definitely deferred, i.e., ESD cannot occur before one lifetime, and so permits informed state preparation with the initial entanglement and purity values within its borders.

## VI. SUMMARY

In summary, we have reported new results bearing on the behavior of non-local decoherence and its potential for being managed or even controlled. As is well known and demonstrated [19, 20] ESD exists, i.e., decoherence processes can drive prepared entanglement to zero at the same time that easily monitored local coherences and fidelity remain non-zero. Its inverse process, sudden birth of entanglement [21], is equally interesting and is the subject of a later report.

Until now there have been no rules of thumb or intuitive guides giving reliable information about the likely occurrence or non-occurrence of ESD even in the simplest instances of entanglement and under the simplest of decoherence mechanisms. We have considered a Bell superposition state as practically the simplest possible entanglement scenario, and have subjected it to amplitude damping, a well-understood decoherence mechanism. Similar steps have been taken previously but only for specific initial density matrices generated *ad hoc* for demonstration purposes.

In contrast, our findings amount to a first step toward defining a set of management tools that permit the range and extent of entanglement decoherence to be bounded and controlled. These tools, i.e., the values of initial purity, entanglement and excitation, are based only on knowledge of parameters that are presumably under good control during any specific entangled state preparation. More importantly, the zone-control nature of these management tools, i.e., the fact that ESD susceptibility is determined over separate zones of a continuous variation of initial conditions, allows a well-defined amount of margin for errors in practical imperfect preparations. Notably, knowledge or control of initial phases is not needed.

## VII. ACKNOWLEDGEMENT

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